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REPLACEMENT POLICIES IN A BIVARIATE FAILURE MODEL.(U)

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CLEMSON UNIVERSITY
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REPLACEMENT POLICIES IN A BIVARIATE FAILURE MODEL

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REPLACEMENT POLICIES IN A BIVARIATE FAILURE MODEL

by

Thomas J. Tosch
and
Paul T. Holmes

ABSTRACT: A two-component failure system in which failure of the secondary component affects the residual life of the primary component is discussed. From two classes of policies and two criteria, optimal policies for replacing the secondary component are derived.

1. Introduction

The bivariate failure model considered is a special case of one used by Tosch and Holmes [2] (see also [3]). Of the two components, one is considered essential to the functioning of the system and is called the major component. The other component is minor in that the system can function without it. The residual lifetime of the major component depends on whether the minor component is functioning or failed. At the failure of the minor component a decision is made whether to replace it or not. This decision is based upon the particular replacement policy being used. It is the aim of this paper to find policies that are optimal in some sense.

In section 2, we formally define the model and give an example of its applicability. Further, the replacement policy classes P and Π and the criteria Γ_1 and Γ_2 for comparing policies are defined. In sections 3 and 4, details of the calculations needed to find optimal policies are given for the classes P and Π respectively. A numerical example is included. The mean residual lifetime function, $m_I(t)$ is discussed by examples in section 5. Some extensions of the results of section 4 are discussed in section 6.

2. Definition of the Model and Example

Label the two components of a system by A and B and let their lifetimes be given by S and T respectively. We let A be the major or primary component of the system. B is the minor or secondary component. This means that the system functions if and only if A functions. We assume that the residual lifetime of A depends on whether B is functioning or not. If B never failed then S=X, where X is a given random variable. At the failure of B, however, A has a residual lifetime distributed as U, another given random variable. It is further assumed that X,U and T are mutually independent, positive random variables with finite means. A generalization of this model which includes a change in the residual life of B at a failure of A is discussed in Tosch and Holmes [2].

When B fails, it can be replaced instantaneously so that there is no effect on the residual lifetime of A. If B is not replaced, then the residual lifetime of A is U. When A fails, the system fails and all failed components are replaced. The

system then begins functioning again.

Let $S_1, S_2, \ldots.$ be the failure times of the system. Let the minor component have an exponentially distributed lifetime, that is, $T \sim \exp(\beta)$. At the failure of A, if B is working, the residual lifetime of B is exponentially distributed with parameter β. Thus $\{S_1, S_2, \ldots.\}$ is a renewal process. Let S be a random variable with the same distribution as the inter-renewal times $S_1, S_2 - S_1, \ldots.$ Under any replacement policy we have

$$E(S) \leq E(X)+E(U) < \infty \qquad (2.1)$$

The inter-renewal periods will be called cycles.

As an application of this model, let A and B be two electrical generators supplying a hospital. Suppose A supplies 3/4 of the electricity while B only supplies 1/4. When generator B fails, A is powerful enough to supply all of the power but with added risk of failure itself. B, however, is unable to supply all of the power needed, so that A must be repaired whenever it fails. When generator B fails, we make the decision either to fix it immediately or wait until A fails to repair them both.

The following notation will be used:

c--cost of replacing component B

K--cost of replacing component A

(2.2)

(R;P)--the length of the interval R using replacement policy P

C(R;P)--the total costs incurred during interval R using policy P.

R may be random or fixed. We now introduce the criteria and

replacement policices to be considered.

A natural criterion to use is that of minimizing the stationary average costs per unit time. That is, minimize

$$\lim_{t\to\infty} \frac{EC((0,t];P)}{t},$$

where P comes from a particular class of policies.

Since $\{S_1, S_2, \ldots\}$ is a renewal process and $E(S) < \infty$, by a theorem due to Johns and Miller (c.f., Ross[1], p. 52), we have

$$\lim_{t\to\infty} \frac{EC((0,t];P)}{t} = \frac{EC((0,S];P)}{E((0,S];P)} . \tag{2.3}$$

This is the basis for the following criterion.

Definition 1. Let Γ_1 be the criterion which judges policy P better than policy Q if and only if

$$\frac{EC((0,S];P)}{E((0,S];P)} < \frac{EC((0,S];Q)}{E((0,S];Q)} .$$

It is also reasonable to consider money already spent as gone and only consider future costs. In particular, consider only the costs from time t, a failure time of B, until the end of the present cycle.

Definition 2. Let Γ_2 be the criterion which judges policy P better than policy Q if and only if

$$\frac{EC([t;S];P)}{E([t;S];P)} < \frac{EC([t;S];Q}{E([t;S];Q)} ,$$

given a failure of B at time t.

We will now define the policies to be considered.

Definition 3. Let P_t be the replacement policy by which B is replaced anytime it fails before fixed time t, measureed from the beginning of the cycle. It is not replaced if a failure occurs after time t. Let P be the class of such policies.

Each time B fails, a decision is made whether to replace it or not. Of course, once a decision is made not to replace B, there are no more decisions to be made in that cycle. The criteria selected require the calculation of future costs in the cycle. Therefore to judge a policy we have to specify what actions will be taken in the future of the cycle and not just at a failure time of B.

Definition 4. Let Π be the class of policies where, given a failure of B at time t, our decision whether to replace B or not is based on a comparison between the policies of not replacing B and continuing to replace B until A fails.

3. Optimizing in class P with criterion Γ_1

We seek to find $t \geq 0$ so that $\frac{EC((0,S];P_t)}{E((0,S];P_t)}$ is a minimum. Let

$$d(t)=E((0,S];P_t)=\int_0^\infty E((0,S];P_t|X=x)dF_X(x)$$

$$=\int_0^t E((0,S];P_t|X=x)dF_X(x)+\int_{t+}^\infty E((0,S];P_t|X=x)dF_X(x). \tag{3.1}$$

On the interval $[0,t]$, $E((0,S];P_t|X=x)=x$, since A fails before we would have stopped replacing B. On (t,∞), we have

$$\int_{t+}^{\infty} E((0,S];P_t|X=x)\,dF_X(x) = \int_0^{\infty}\int_{t+}^{\infty} E((0,S];P_t|X=x,T=y)\,dF_X(x)\,\beta e^{-\beta y}dy$$
$$= \int_0^{\infty}\int_{t+}^{t+y} E((0,S];P_t|X=x,T=y)\,dF_X(x)\;e^{-\beta y}dy$$
$$+\int_0^{\infty}\int_{t+y}^{\infty} E((0,S];P_t|X=x,T=y)\,dF_X(x)\,\beta e^{-\beta y}dy \qquad (3.2)$$

In both of these integrals $X > t$, so that at time t both components are functioning and the residual lifetime of B is exponentially distributed with parameter β. We condition on this lifetime so that B would fail at time t+y. In the first integral of (3.2) we have the relation $t < x < t+y$ which means A fails sooner after t than B would have. This implies

$$\int_{t+}^{\infty} E((0,S];P_t|X=x)\,dF_X(x)$$
$$=\int_0^{\infty}\int_{t+}^{t+y} x\,dF_X(x)\,\beta e^{-\beta y}dy+\int_0^{\infty}\int_{t+y}^{\infty}(t+y+E(U))\,dF_X(x)\,\beta e^{-\beta y}dy$$
$$=\int_t^{\infty}\int_{x-t}^{\infty}\beta e^{-\beta y}dy\,x\,dF_X(x)+\int_t^{\infty}\int_0^{x-t}(t+y+E(U))\,\beta e^{-\beta y}dy\,dF_X(x)$$
$$=\int_t^{\infty} e^{-\beta(x-t)}x\,dF_X(x)+\int_t^{\infty}\left[(t+E(U))(1-e^{-\beta(x-t)})+\frac{1}{\beta}-e^{-\beta(x-t)}(x-t+\frac{1}{\beta})\right]dF_X(x)$$
$$=\int_t^{\infty}\left\{e^{-\beta(x-t)}\left[x-t-E(U)-x+t-\frac{1}{\beta}\right]+t+E(U)+\frac{1}{\beta}\right\}dF_X(x)$$
$$=(t+E(U)+\frac{1}{\beta})\overline{F}x(t)-(E(U)+\frac{1}{\beta})\int_t^{\infty}e^{-\beta(x-t)}dF_X(x).$$

Substituting these quantities into equation (3.1) to obtain an expression for d(t), we have

$$d(t)=E((0,S];P_t)=\int_0^t x\,dF_x(x)+(t+E(U)+\tfrac{1}{\beta})\overline{F}_x(t)$$
$$-(E(U)+\tfrac{1}{\beta})\int_t^\infty e^{-\beta(x-t)}dF_x(x). \qquad (3.3)$$

Let $n(t)=EC((0,S];P_t)=\int_0^\infty EC((0,S];P_t|X=x)dF_x(x)$

$$=\int_0^t EC((0,S];P_t|X=x)dF_x(x)+\int_{t+}^\infty EC((0,S];P_t|X=x)dF_x(x). \qquad (3.4)$$

On the interval [0,t], $EC((0,S];P_t|X=x)=c\beta x+K$ since x is the length of the cycle and βx is the expected number of times that B had to be replaced. The second integral in (3.4) is handled as above in the derivation of d(t). After simplifying we have

$$n(t)=c(\beta t+1)\overline{F}_x(t)+K+c\beta\int_0^t x\,dF_x(x)-c\int_t^\infty e^{-\beta(x-t)}dF_x(x). \qquad (3.5)$$

We wish to minimize $\frac{h(t)}{d(t)}$ for $t \geq 0$. $\frac{h(t)}{d(t)}$ is differentiable, so that the global minimum will occur at the extremes or at a time t where the derivative is zero. The values at the extremes are given by

$$\frac{h(0)}{d(0)} = \frac{K+c(1-f_x^*(\beta))}{(E(U)+\frac{1}{\beta})\ \ (1-f_x^*(\beta))}\ , \qquad (3.6)$$

where f_x^* is the Laplace-Stieltjes transform of F_x, and

$$\lim \frac{n(t)}{d(t)} = \frac{K+c\beta E(x)}{E(x)}\ . \qquad (3.7)$$

The numerator of the derivative controls the sign. Let r(t) be the numerator of the derivative, then it can be shown

that:

$$r(t)=c\beta E(U)\,[\overline{F}_x^2(t)+(\beta\int_0^t x\,dF_x(x)-\overline{F}_x(t)+\beta E(U)\,t\overline{F}_x(t))\int_t^\infty e^{-\beta(x-t)}dF_x(x)]$$

$$-K[\overline{F}_x(t)-(\beta E(U)+1)\int_t^\infty e^{-\beta(x-t)}dF_x(x)]. \quad (3.8)$$

This expression is difficult to work with in general, but may be evaluated, at least numerically, for particular applications.

Example 1. Consider the application in the beginning of the report. Let $T\sim\exp(4)$, $X\sim\exp(2)$ and $c=1000$, $K=3000$, $E(U)=.25$. The graph of $\frac{n(t)}{d(t)}$, $0 \leq t \leq 3$ is given below.

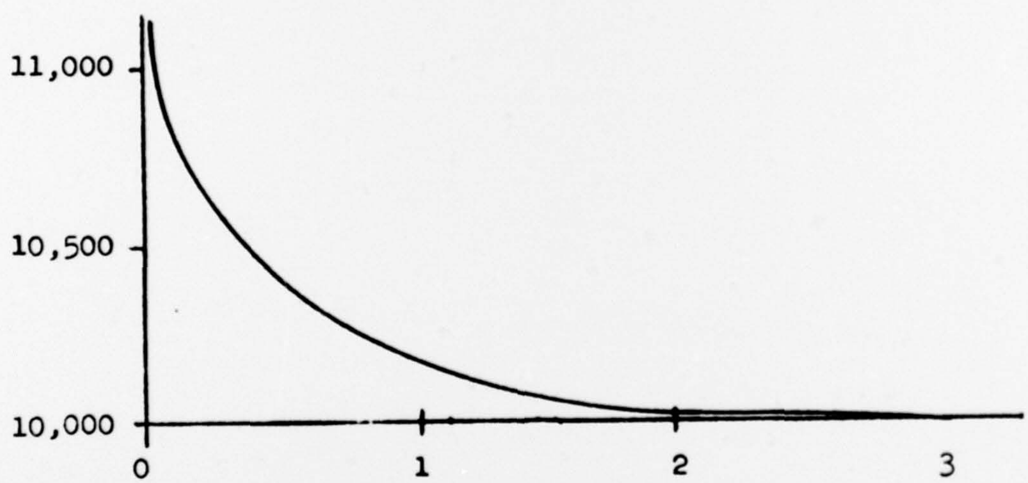

Figure 1. A graph of n(t)/d(t).

In this example, $\frac{n(t)}{d(t)}$ is monotone decreasing, so that the optimal policy is to always replace B when it fails.

The difficulty in this derivation probably lies in the choice of the class of policies. We are assuming that if it is not optimal to replace B at time $t+s_1$, then it is not optimal to replace B had it failed instead at $t+s_2$, where $s_2 > s_1 > 0$. If the failure rate function of X, $h_X(t)=f_X(t)/\overline{F}_X(t)$, is like that in Figure 2, that assumption would certainly not be justified.

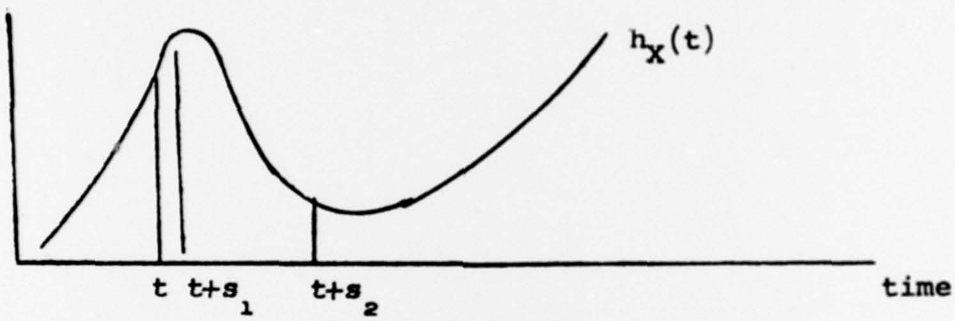

Figure 2. Example of the Function $h_X(t)$.

Here it may not be optimal to replace B if it fails at time $t+s_1$, since A is approaching a very critical period where it is likely to fail anyway. At time $t+s_2$, however, component A is very reliable and therefore it may be optimal to replace B at that time.

4. Optimizing in the class Π

Throughout the rest of the paper, we assume that a failure of B has occurred at time t. All probabilistic statements are conditioned on that assumption. The class Π and criterion Γ_2 will be considered first. At time t, component B has failed and

we must decide whether to replace B or not. We won't replace B if and only if the policy of not replacing B is "better" than always replacing B until A fails. "Better" here depends on the criterion being used, in this case Γ_2. Let Π_0 be policy of not replacing B at time t, and Π_1 be the policy of replacing B until A fails. The optimal decision will be to replace B at time t if and only if

$$\frac{EC([t;S);\Pi_1)}{E([t;S];\Pi_1)} < \frac{EC([t,S];\Pi_0)}{E([t,S];\Pi_0)} \tag{4.1}$$

Under Π_0, S-t=u so that

$$E([t,S];\Pi_0)=E(U). \tag{4.2}$$

The costs involved are those of replacing B and A once each so that

$$EC([t,S];\Pi_0)=c+K \tag{4.3}$$

Under Π_1 the residual length of the cycle is the residual length of X after t. The expected value of this length is

$$E([t,S];\Pi_1)=m_x(t)=\frac{\int_t^{\infty}\overline{F}_x(x)\,dx}{\overline{F}_x(t)}\,. \tag{4.4}$$

Finally, the expected costs incurred during this period are $c+K+c\beta m_x(t)$, the last term enters in since $\beta m_x(t)$ is the expected number of times we will have to replace B in the interval $(t,S]$. Therefore

$$EC([t,S],\Pi_1)=c+K+c\beta m_x(t) \tag{4.5}$$

Combining these into (4.1) and doing the algebra, we obtain

Theorem 6: In class Π with criterion Γ_2, it is optimal to replace a failed component B at time t if and only if

$$(c+K-c\beta E(U))m_x(t) > (c+K)E(U).$$

Since $m_x(t)$ is not monotone in general, these policies will not have the cutoff point between replacing and not replacing as did the policies presented in the previous section. The following corollary is immediate.

Corollary 7: If $c\beta E(U) > c+K$ then it is optimal never to replace component B.

The condition $c\beta E(U) > c+K$ is equivalent to $[\frac{E(U)}{E(T)} - 1] > \frac{K}{c}$. Heuristically this says that E(U), the expected lifetime of A without B, is so much larger than E(T), the lifetime of B, that it is not worth the risk of c dollars to even have a component B in the system. Under normal circumstances then, this condition will not be satisfied.

If $X \sim \exp(\alpha)$, then $m_x(t)=\frac{1}{\alpha}$ which implies

<u>Corollary 8</u>: If $X \sim \exp(\alpha)$, then in class Π with criterion Γ_2, it is optimal to

i) never replace B, if $(c+K-\beta E(U)) \leq \alpha(c+K)E(U)$,

ii) always replace B, if $(c+K-\beta E(U)) > \alpha(c+K)E(U)$.

This result is very intuitive since at any failure of B, the future looks exactly the same as at any other failure of B. This implies that the same decision should be made at each failure of B.

In example 5, we have $(c+K-c\beta E(U))\frac{1}{\alpha} = (3000)\frac{1}{2} = 1500$, while $(K+c)E(U)=(4000)(.25)=1000$. So that in Π under criterion Γ_2, it is optimal to always replace B when it fails.

Basing the criterion only on future costs has the drawback that K, the cost of replacing component A, is distributed only on the future time in the cycle. For this reason, we will consider the same class Π but with criterion Γ_1. Let Π_0,Π_1 be as before. Under Γ_1, it is optimal to replace B at time t if and only if

$$\frac{EC((0,S[;\Pi_1)}{E((0,S];\Pi_1)} < \frac{EC((0,S];\Pi_0)}{E((0,S];\Pi_0} \tag{4.6}$$

As before it is easy to obtain expressions for these quantities. We have

$$E((0,S];\Pi_0)=t+E(U) \tag{4.7}$$

$$E((0,S];\Pi_1)=t+m_x(t). \tag{4.8}$$

These follow since the cycle is already t units old when we make our decision. The expected number of times we have replaced B in (0,t) is βt at a cost of $c\beta t$ so that the other quantities can be written as:

$$EC((0,S];\Pi_0)=c\beta t+c+K, \tag{4.9}$$

$$EC((0,S];\Pi_1)=c\beta t+c+K+c\beta m_x(t). \tag{4.10}$$

Combining these into (4.6), our decision is to replace if

$$\frac{c\beta t+c+K+c\beta m_x(t)}{t+m_x(t)} < \frac{c\beta t+c+K}{t+E(U)}, \tag{4.11}$$

from which we get

<u>Theorem 9</u>: In class Π with criterion Γ_1, it is optimal to replace component B at time t if and only if

$$(K+c-c\beta E(U))m_x(t) > c\beta E(U)t+(K+c)E(U).$$

<u>Corollary 10</u>: If $K+c < c\beta E(U)$ then it is optimal to never replace B.

The proof is trivial. The remarks on the condition following corollary 7 are valid here also.

Returning to example 5, $(c+K-c\beta E(U))m_x(t)=1500$ for all $t \geq 0$. $(K+c)E(U)=1000$ and $c\beta E(U)=1000$. Therefore it is optimal

under Γ_1, to replace B if and only if $1500 > 1000t+1000$, i.e., if t is in the interval $[0,\frac{1}{2})$.

There is a large similarity in the conditions needed for replacement under Γ_1 and Γ_2. In fact , we have

Corollary 11: If the optimal decision under Γ_1 is to replace B then it is also the optimal decision under Γ_2.

The converse of the corollary is clearly not true as example 5 illustrates.

With criterion Γ_1 then, it is optimal to replace B when the mean residual lifetime of A is greater than the given linear function. The next section includes graphs of mean residual lifetime functions for various distributions to illustrate the possible policies.

5. The Function $m_I(t)$

The following figures illustrate the various forms of $m_I(t)$.

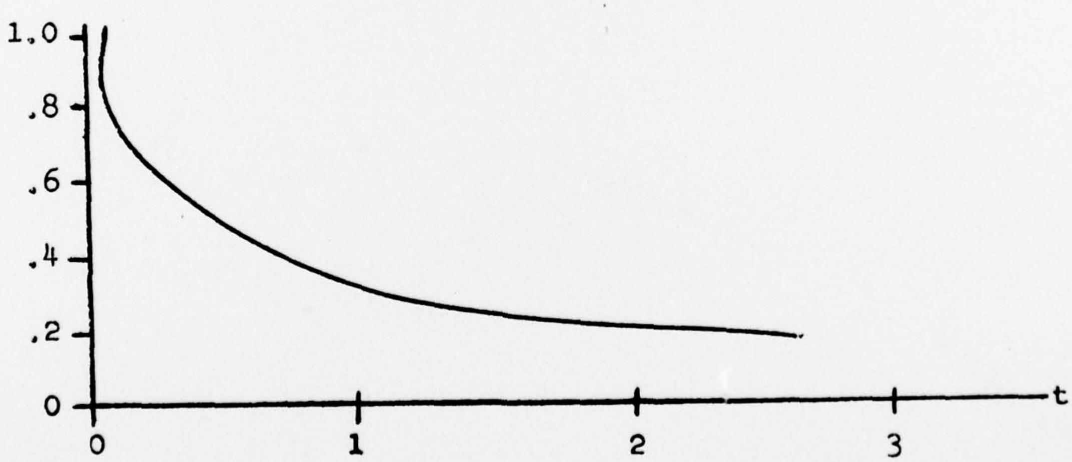

Figure 3. $m_I(t)$ for a Weibull Distribution with $\beta > 1$

$$\bar{F}_I(t)=e^{-t^2} .$$

In this case there will be a unique time t*, where we will replace B if it fails before t*, but not afterwards.

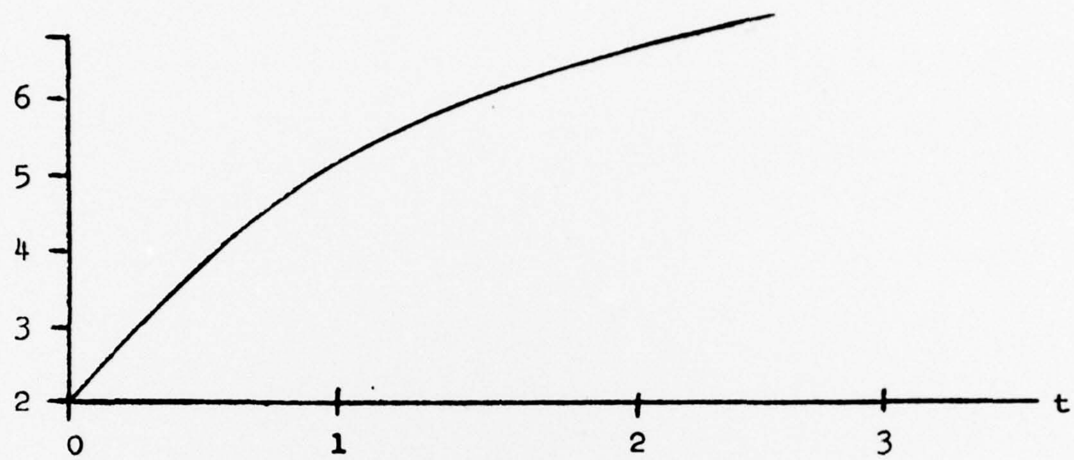

Figure 4. $m_I(t)$ for a Weibull Distribution with $\beta < 1$

$$\bar{F}_I(t)=e^{-\sqrt{t}}$$

In addition to the previous policy, in this case, there could be t_1,t_2 so that if B fails in (t_1,t_2) we will replace it; otherwise we will not.

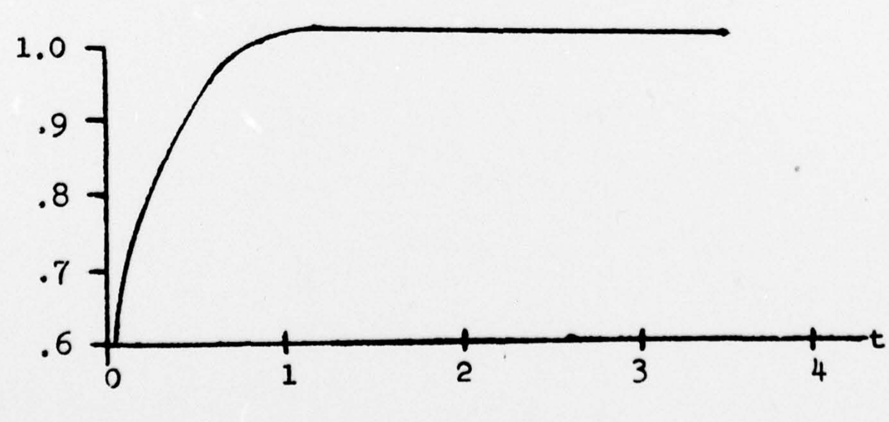

Figure 5. $m_I(t)$ for a Mixture of Exponentials

$$\bar{F}_I(t)=.5e^{-5t}+.5e^{-t} .$$

The same possible optimal policies exist here as in the previous example.

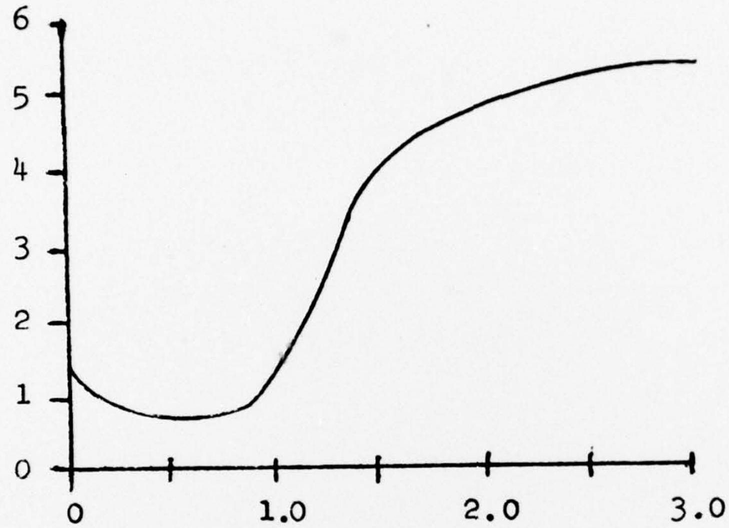

Figure 6. $m_I(t)$ for a Mixture of Weibull Distributions

$$\bar{F}_I(t)=0.7e^{-t^4}+0.3e^{-\sqrt{t}}$$

Depending on the given constants, there are a variety of optimal policies that could occur here. One possibility is that there exists t_1,t_2,t_3 so that we replace on the intervals $[0,t_1)$, (t_2,t_3) and do not otherwise.

6. Extensions

The costs that have been considered are very simple. What has been done is easily extended to more complicated costs. The

extensions will be applied to optimizing in the class Π under criterion Γ_1.

Suppose that the components are not replaced but are repaired. Let K be a random variable which gives the repair cost of component A, and Δ be the random repair cost of B. If we let $K=E(K)$ and $c=E(\Delta)$, then the analysis is exactly as before and the optimal policy is given by Theorem 9.

Suppose instead that the system has to be stopped in order to make any replacements, and that a penalty cost of δ is charged for each unit of time that system is not functioning. Let H_1 be the random length of time needed to replace A and H_2, the time needed to replace B. Assume that the components cannot be replaced simultaneously. Let $c'=c+\gamma E(H_2)$, $K'=k+\gamma E(H_1)$. The analysis goes as before and the optimal policy is given by Theorem 9 with K',c' substituted for K,c.

If the components can be replaced simultaneously then there is a change, since if we did not replace B when it failed, A and B could be replaced at the same time. In this case, it is optimal to replace B at time t if and only if

$$\frac{c'\beta t+c'+K'+c'\beta m_x(t)}{t+m_x(t)} < \frac{c'\beta t+c+K+\gamma E(\max(H_1,H_2))}{t+E(U)} \tag{6.1}$$

The last extension to be discussed deals with the random variable U. It has been assumed that U and X are independent. This is often not even a good approximation. Suppose U is

dependent on the residual lifetime of X, in that, $E(U)=\varepsilon m_x(t)+\delta$, $\varepsilon \geq 0, \delta \geq 0$. In this case, it is optimal to replace B at time t if and only if

$$\frac{c\beta t+c+k+c\beta m_x(t)}{t+m_x(t)} < \frac{c\beta t+c+K}{t+\varepsilon m_x(t)+\delta} . \tag{6.2}$$

This reduces to the condition

$$c\beta\varepsilon m_x^2(t)+m_x(t)[c\beta\varepsilon t+(K+c)\varepsilon-(K+c)+c\beta\delta]+c\beta\delta t+(K+c)\delta < 0. \tag{6.3}$$

So in this case, there is a quadratic relationship between t and $m_x(t)$. Note $\varepsilon=0$ corresponds to the case in section 4.

Other extensions can be made with little change in the form of the optimal policy. If we do not restrict T to being exponential then we must include repairing B when A fails into our model to retain the renewal process.

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